Note di Matematica
Note Mat. 1(2008), 29-42
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1p29
http://siba2.unile.it/notemat

Spectral geometry for the Jacobi operator of the identity map

Luigi Vergori
Università del Salento, Departimento di Matematica,
Strada provinciale Lecce-Arnesano, 73100 Lecce, Italy
luigi.vergori@unile.it

Received: 06/09/2006; accepted: 01/12/2006.

Abstract. Let (M, g) be a compact Riemannian manifold. We denote by J_M the Jacobi operator of the identity map which is a second order elliptic differential operator. In this paper we study the following problem: which geometric properties are determined by the spectrum $\mathrm{Spec}(J_M)$?

Keywords: harmonic map, identity map, isospectrality, Jacobi operator

MSC 2000 classification: primary 58J50–58J53, secondary 53C25–53C45

Introduction

Let (M, g) and (N, h) be two Riemannian manifolds, (M, g) being compact. The Jacobi operator J_f (also called second variation operator) of a harmonic map $f : (M, g) \to (N, h)$ is a second order elliptic differential operator. H. Urakawa [14], applying the Gilkey results [4] to the Jacobi operator J_f, obtained a series of interesting geometric results distinguishing typical harmonic maps.

The identity map I_M is a trivial example of harmonic map but the theory of the corresponding Jacobi operator J_M is much more complex.
If (M, g) and (M', g') are two compact isometric Riemannian manifolds, it is well-known that $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$. As the inverse problem is concerned: Are (M, g) and (M', g') isometric if $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$?
The answer is, in general, negative. In fact, if (T^n, g) is a flat torus, then $\mathrm{Spec}(J_M) = n \times \mathrm{Spec}(\Delta)$ ([14], page 258), where Δ is the Laplace-Beltrami operator, and Milnor (see [7] or [2], page 154) showed that there exist two 16-dimensional flat tori which are isospectral (with respect to Δ) but not isometric. However the spectrum of J_M is strictly linked to the geometry and, in particular, to the curvature of (M, g) [14, 11, 5]. Therefore it is reasonable to investigate the geometrical properties which can be derived from the spectrum and, in particular, if special Riemannian manifolds (M, g) can be characterized by the spectrum $\mathrm{Spec}(J_M)$. The analogous problem for the Jacobi operator of a Riemannian foliation has been studied by Nishikawa, Tondeur and Vanhecke in [8].

The main aim of this paper consists in characterizing the Euclidean sphere S^n by the spectrum $\mathrm{Spec}(J_{S^n})$ and the corresponding problems for the complex projective space $\mathbb{CP}^{2q}$ and the quaternionic projective space $\mathbb{HP}^{4q}$ endowed with the standard metrics.

1 Preliminaries

In this section, we apply the Gilkey results [4] to the Jacobi operator of the identity map of a compact Riemannan manifold. First we recall the second variation formula of the energy of a harmonic map.

Let (M,g) be a n-dimensional compact Riemannian manifold and (N,h) a m-dimensional Riemannian manifold. A smooth map $\phi:(M,g)\to(N,h)$ is said to be harmonic if it is a critical point of the functional energy $E(\cdot)$

$$E:\phi\in C^\infty(M)\mapsto \frac{1}{2}\int_M \mathrm{tr}_g(\phi^* h)v_g\in\mathbb{R},$$

where ϕ^* is the dual map of the differential of ϕ and v_g is the volume element of (M,g), namely, for any vector field V along ϕ:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)\bigg|_{t=0}=0.$$

Here $\phi_t:M\to N$ is a one parameter family of smooth maps with $\phi_0=\phi$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(p)\bigg|_{t=0}=V_p\in T_{\phi(p)}N\quad\forall p\in M.$$

The second variation formula of the energy E for a harmonic map ϕ is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(\phi_t)\bigg|_{t=0}=\int_M h(V,J_\phi V)v_g.$$

The differential operator J_ϕ, called the Jacobi operator of ϕ, is given by

$$J_\phi V=\bar\Delta_\phi V-Ric_\phi V,$$

where $\bar\Delta_\phi$ is the rough Laplacian along ϕ and, denoting by R_h the curvature tensor of (N,h) and by $\{e_i\}_{i=1,\dots,n}$ a local orthonormal frame field on (M,g),

$$Ric_\phi V=\sum_{i=1}^n R_h(\phi_* e_i,V)\phi_* e_i.$$

Since J_ϕ is self-adjoint and elliptic and M is compact, J_ϕ has a discrete spectrum of real eigenvalues with finite multiplicities:

$$\operatorname{Spec}(J_\phi) = \{\mu_1 \le \mu_2 \le \cdots \le \mu_i \le \cdots \uparrow +\infty\}.$$

ϕ is said to be *stable* if $\mu_1 \ge 0$, *unstable* if $\mu_1 < 0$. The trace

$$Z(t) = \sum_{i=1}^{+\infty} \exp(-t\mu_i)$$

of the heat kernel for the operator J_ϕ has the asymptotic expansion

$$Z(t) \sim \frac{1}{(4\pi t)^{n/2}} \left\{ a_0(J_\phi) + a_1(J_\phi)t + a_2(J_\phi)t^2 + \cdots \right\} \quad \text{as } t \to \infty. \tag{1}$$

According to [4, 14],

$$a_0(J_\phi) = m \operatorname{vol}(M, g), \tag{2}$$

$$a_1(J_\phi) = \frac{m}{6} \int_M r v_g + \int_M \operatorname{tr}_g(\phi^* \rho_h) v_g \tag{3}$$

and

$$\begin{aligned} a_2(J_\phi) = {} & \frac{m}{360} \int_M \left[5r^2 - 2\|\rho\|^2 + 2\|R\|^2 \right] v_g \\ & + \frac{1}{360} \int_M \left[-30\|\phi^* R_h\|^2 + 60r \operatorname{tr}_g(\phi^* \rho_h) + 180\|Ric_\phi\|^2 \right] v_g, \end{aligned} \tag{4}$$

where R, ρ, r denote respectively the curvature tensor, the Ricci tensor, the scalar curvature of (M, g) and ρ_h is the Ricci tensor of (N, h). For all X, Y vector fields belonging to $\mathcal{X}(M)$ and for all V vector field along ϕ we set $(\phi^* R_h)(X, Y)V = R_h(\phi_* X, \phi_* Y)V$.

If we consider the identity map I_M, which is a harmonic map, and denote by J_M the corresponding Jacobi operator, then the spectral invariants (2), (3) and (4) are now given by

$$a_0(J_M) = n \operatorname{vol}(M, g), \tag{5}$$

$$a_1(J_M) = \left(\frac{n}{6} + 1 \right) \int_M r v_g, \tag{6}$$

and

$$a_2(J_M) = \frac{1}{360} \int_M \left[5(n+12)r^2 + 2(90-n)\|\rho\|^2 + 2(n-15)\|R\|^2 \right] v_g \tag{7}$$

From (1), (5)-(7) we readily get

1 Theorem. *Let* (M, g) *and* (M', g') *be two compact Riemannian manifolds. If* $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, *then*

$$n = \dim M = \dim M' = n', \tag{8}$$

$$\operatorname{vol}(M, g) = \operatorname{vol}(M', g'), \tag{9}$$

$$\int_M r v_g = \int_{M'} r' v'_g, \tag{10}$$

$$\begin{aligned}\int_M &\left[5(n+12)r^2 + 2(90-n)\|\rho\|^2 + 2(n-15)\|R\|^2\right] v_g \\ &= \int_{M'} \left[5(n+12)r'^2 + 2(90-n)\|\rho'\|^2 + 2(n-15)\|R'\|^2\right] v'_g. \end{aligned}\tag{11}$$

2 Spectral geometry of J_M for Riemannan manifolds

Let (M, g) be a n-dimensional Riemannian manifold. We will now use the spectral invariants (8)-(11) to derive some results about the geometry of M. Before starting on this we recall that, for $n \geq 3$,

$$\|\rho\|^2 \geq \frac{r^2}{n} \tag{12}$$

where the equality holds if and only if (M, g) is an Einstein manifold. Further

$$\|R\|^2 \geq \frac{2}{n(n-1)} r^2 \tag{13}$$

where the equality holds if and only if (M, g) has constant sectional curvature and $n \geq 3$. For $n = 2$ we always have the equality.

Next, for $n \geq 3$, we denote by C the Weyl conformal curvature tensor associated to R. Then (see for example [8]):

$$\|C\|^2 = \|R\|^2 - \frac{4}{n-2}\|\rho\|^2 + \frac{2}{(n-1)(n-2)} r^2. \tag{14}$$

For $n = 3$, $C = 0$ while for $n \geq 4$, $C = 0$ if and only if the Riemannian manifold (M, g) is conformally flat. Finally we note that, for $n \geq 4$, (M, g) has constant sectional curvature if and only if it is conformally flat and Einstein.

In the sequel we denote by n the dimension of M and by n' the dimension of M'.

Now we are ready to prove the following results.

2 Theorem. *Let (M,g) and (M',g') be two compact Riemannian manifolds with $n \in \{2,3,16,17,18,\dots,93\}$. If* $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, *then (M',g') has constant sectional curvature c' if and only if (M,g) has constant sectional curvature $c = c'$.*

Proof. We suppose that (M',g') has constant sectional curvature c' and observe that from (8) $n' = \dim M' = \dim M = n$.

We assume first $16 \le n \le 93$. From (14) and (11) we get

$$\int_M \left[\alpha_n \|C\|^2 + \beta_n \left(\|\rho\|^2 - \frac{r^2}{n}\right)\right] v_g = \gamma_n \left(\int_{M'} r'^2 v_{g'} - \int_M r^2 v_g\right) \tag{15}$$

with

$$\alpha_n = 2(n-15),$$
$$\beta_n = \frac{-2n^2 + 192n - 480}{n-2}$$

and

$$\gamma_n = \frac{5n^4 + 43n^3 + 20n^2 - 492n + 480}{n(n-1)(n-2)} = \frac{5n^3 + 45n^2 - 174n + 180}{(n-1)(n-2)} + \frac{\beta_n}{n}.$$

It is easy to check that α_n, β_n and γ_n are positive for $16 \le n \le 93$.

Since r' is constant, $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$ implies

$$\int_M r^2 v_g \ge \int_{M'} r'^2 v_{g'}. \tag{16}$$

In fact, by using (9) and (10), we have

$$\begin{aligned} \int_M r^2 v_g - \int_{M'} r'^2 v_{g'} &= \int_M r^2 v_g - 2r' \int_{M'} r' v_{g'} + r'^2 \int_{M'} v_{g'} \\ &= \int_M \left(r^2 - 2rr' + r'^2\right) v_g = \int_M (r - r')^2 v_g \ge 0. \end{aligned} \tag{17}$$

From (17) we readily deduce that the equality sign in (16) holds if and only if $r = \mathit{constant} = r'$. Hence (15) and (16) yield $C = 0$ and $\|\rho\|^2 = r^2/n$. So (M,g) has constant sectional curvature c. This is equal to c' because $r = r' = n(n-1)c'$.

For $n = 3$, we have $C = 0$ and hence (14) and (11) give

$$78 \int_M \left(\|\rho\|^2 - \frac{r^2}{3}\right) v_g = 125 \left(\int_{M'} r'^2 v_{g'} - \int_M r^2 v_g\right).$$

Then (16) implies that $\|\rho\|^2 = r^2/3$ and $r = r'$. Consequently (M,g) is a 3-dimensional Einstein manifold and so it has constant sectional curvature c. This equals c' since $r = r'$.

For $n = 2$, denoting by K and K' the Gaussian curvatures of M and M', respectively, we have

$$r = 2K, \quad \|R\|^2 = 4K^2, \quad \|\rho\|^2 = 2K^2.$$

Then, from (11), we get

$$\int_M r^2 v_g = 4\int_M K^2 v_g = 4\int_{M'} K'^2 v_{g'} = \int_{M'} r'^2 v_{g'}$$

So, from (16) and (17) we have $K = constant = K' = c'$. $\boxed{QED}$

We now denote by $(S^n(c), g_0)$ the Euclidean n-sphere with constant sectional curvature $c > 0$. Theorem 2 gives the following

3 Corollary. *Let (M, g) be a compact simply connected Riemannian manifold. If $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{S^n(c)})$, with $n \in \{2, 3, 16, 17, 18, \dots, 93\}$, then (M, g) is isometric to $(S^n(c), g_0)$.*

4 Proposition. *Let (M, g) and (M', g') be two compact Riemannian manifolds with $n = 2$. If $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, then the Euler numbers $\chi(M)$ and $\chi(M')$ are equal. In particular, if M and M' are both orientable (resp. not orientable), then M and M' are homeomorphic.*

Proof. From (8) $\dim M' = \dim M = 2$, so

$$r = 2K \quad \text{and} \quad r' = 2K'.$$

Then, by Gauss-Bonnet formula and (10), we get

$$\chi(M) = \frac{1}{2\pi}\int_M K v_g = \frac{1}{4\pi}\int_M r v_g = \frac{1}{4\pi}\int_{M'} r' v'_g = \frac{1}{2\pi}\int_{M'} K' v'_g = \chi(M').$$

If M and M' are both orientable (resp. not orientable) then they have the same genus $p = 1 - \chi(M)/2$ (resp. $q = 2 - \chi(M)$) and, by the classification Theorem for 2-dimensional compact connected manifolds [3], they are both homeomorphic to

$$S^2\sharp\underbrace{T^2\sharp\dots\sharp T^2}_{p-times} \quad (\text{resp. } S^2\sharp\underbrace{\mathbb{P}^2\sharp\dots\sharp\mathbb{P}^2}_{q-times}),$$

where $\sharp$, T^2, $\mathbb{P}^2$ denote respectively the connected sum between manifolds, the 2-dimensional torus and the real projective plane. $\boxed{QED}$

5 Proposition. *Let (M, g) and (M', g') be two compact Riemannian manifolds with $n = 4$ and (M', g') Einstein. If $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, then*

$$\chi(M) \geq \chi(M'), \tag{18}$$

where the equality holds if and only if (M, g) is also an Einstein manifold and $r = r'$.

Proof. We start noticing that, by (8), $n = n' = 4$. The Gauss-Bonnet formula for any 4-dimensional compact manifold M is given by [3]

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left(\|R\|^2 - 4\|\rho\|^2 + r^2\right) v_g. \tag{19}$$

Since (M', g') is Einstein, by using (11) and (19), (15) becomes

$$\begin{aligned} 704\pi^2\left(\chi(M) - \chi(M')\right) = 84\int_M \left(\|\rho\|^2 - \frac{r^2}{4}\right) v_g \\ + 123\left(\int_M r^2 v_g - \int_{M'} r'^2 v_{g'}\right). \end{aligned} \tag{20}$$

Hence, (12) and (16) give (18).

In particular, if $\chi(M) = \chi(M')$, (20) gives $\|\rho\|^2 = r^2/4$ and $r = r'$. Vice versa if (M, g) is Einstein, the right hand side in (20) vanishes and then the required result follows. $\boxed{QED}$

6 Theorem. *Let (M, g) and (M', g') be two compact Einstein manifolds with $n \neq 15$. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M', g') has constant sectional curvature c' if and only if (M, g) has constant sectional curvature $c = c'$.*

Proof. Since M and M' are both Einstein, using (8)-(10), we get

$$n = n', \quad r = r' \quad \text{and hence} \quad \|\rho\| = \|\rho'\|.$$

From (9) and (11), taking into account that $n \neq 15$, we have

$$\int_M \|R\|^2 v_g = \int_{M'} \|R'\|^2 v_{g'}$$

and then

$$\int_M \left[\|R\|^2 - \frac{2}{n(n-1)} r^2\right] v_g = \int_{M'} \left[\|R'\|^2 - \frac{2}{n(n-1)} r'^2\right] v_{g'}.$$

The required result follows from (13) and the fact that manifolds with constant sectional curvature c satisfy $r = n(n-1)c$. $\boxed{QED}$

By combining Proposition 5 and Theorem 6 we get

7 Theorem. *Let (M, g) and (M', g') be two compact Riemannian manifolds with $n = 4$, $\chi(M) \le \chi(M')$ and (M', g') having constant sectional curvature c'. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M, g) has constant sectional curvature $c = c'$.*

8 Theorem. *Let (M, g) and (M', g') be two compact conformally flat Riemannian manifolds with $2 \le n \le 93$. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M', g') has constant sectional curvature c' if and only if (M, g) has constant sectional curvature $c = c'$.*

Proof. The cases $n = 2$ and $n = 3$ have been studied in Theorem 2. For $4 \leq n \leq 93$ the required result readily follows from (15) and (16) taking into account that $C = C' = 0$ and that β_n and γ_n are positive for such values of n. $\boxed{QED}$

9 Theorem. *Let (M, g) and (M', g') be two compact conformally flat Riemannian manifolds with constant scalar curvature. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M', g') has constant sectional curvature c' if and only if (M, g) has constant sectional curvature $c = c'$.*

Proof. We suppose that (M', g') has constant sectional curvature c'. By (9) and (10) we get $r = r' = n(n-1)c'$. Then (15) implies that (M, g) is Einstein and hence it has constant sectional curvature $c = c'$. $\boxed{QED}$

Finally by Theorems above we get the following

10 Corollary. *Let (M, g) be a compact simply connected Riemannian manifold. Assume that $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{S^n(c)})$. If one of the following cases occurs:*

(a) $n = 4$ and $\chi(M) \leq 2$,

(b) (M, g) Einstein and $n \neq 15$,

(c) (M, g) conformally flat and $2 \leq n \leq 93$,

(d) (M, g) conformally flat with constant scalar curvature and $n \geq 2$,

then (M, g) is isometric to $(S^n(c), g_0)$.

From now on we denote by λ_1 the least positive eigenvalue of the Laplace-Beltrami operator.

11 Proposition. *Let (M, g) and (M', g') be two compact Riemannian manifolds with (M, g) 15-dimensional and Einstein. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then:*

(i) (M', g') is Einstein with $r = r'$,

(ii) $\lambda_1 \geq 2/15r$ if and only if $\lambda'_1 \geq 2/15r$.

Proof. (i) Since (M', g') is Einstein, for $n = 15$, (11) becomes

$$30 \int_M \left(\|\rho\|^2 - \frac{r^2}{15} \right) v_g = 29 \left(\int_{M'} r'^2 v_{g'} - \int_M r^2 v_g \right). \tag{21}$$

By (16) the right hand side in (21) is non-positive and so $\|\rho\|^2 = r^2/15$ and $r = r'$.

(ii) We recall that if (M, g) is a compact Einstein manifold ($\rho = kg$), by Smith stability Theorem [11], the identity map I_M is stable if and only if λ_1 satisfies the inequality $\lambda_1 \geq 2k$. Since (M, g) and (M', g') are Einstein manifolds with $\rho = r/15g$ and $\rho' = r'/15g'$, respectively, Smith stability Theorem and (i) give the required result. $\boxed{QED}$

Now, let (M, g) be a n-dimensional compact Riemannian manifold and assume that there exists $k > 0$ such that $\rho \geq kg$. By Lichnerowicz-Obata Theorem ([2], pages 179-180) λ_1 satisfies the inequality

$$\lambda_1 \geq \frac{n}{n-1}k$$

where the equality holds if and only if (M, g) is isometric to $\left(S^n\left(\frac{k}{n-1}\right), g_0\right)$.

By combining the Lichnerowicz-Obata Theorem and Proposition 11 (i) we achieve the following

12 Corollary. *The Euclidean sphere* $\left(S^{15}(c), g_0\right)$ *is completely characterized by* λ_1 *and* Spec $\left(J_{S^{15}(c)}\right)$.

13 Remark. Let (M, g) be a n-dimensional compact Einstein manifold with $n \geq 3$, $\rho = kg$ and $k > 0$. Assume that I_M is unstable, then, by Lichnerowicz-Obata and Smith Theorems, we have

$$\frac{n}{n-1}k \leq \lambda_1 < 2k.$$

3 Spectral geometry of J_M for Kähler manifolds

Let (M, g, J) be a Kähler manifold with $\dim_{\mathbb{R}} M = n = 2q$, $q = \dim_{\mathbb{C}} M \geq 2$. By B we denote the Bochner curvature tensor associated to R [8]. Then

$$\|B\|^2 = \|R\|^2 - \frac{8}{q+2}\|\rho\|^2 + \frac{2}{(q+1)(q+2)}r^2. \tag{22}$$

Further, we have

$$\|R\|^2 \geq \frac{2}{q(q+1)}r^2 \tag{23}$$

with the equality sign valid if and only if (M, g, J) has constant holomorphic sectional curvature. (M, g, J) is said to be Bochner-Kähler (or Bochner flat) if its conformal curvature Bochner tensor B vanishes. We remark that (M, g, J) is a Bochner-Kähler-Einstein manifold if and only if it has constant holomorphic sectional curvature.

Now let ω be the cohomology class represented by the fundamental 2-form Φ, and c_1 the first Chern class of (M, g, J). It is well-known that c_1 is represented by the Ricci form γ. Following [9] (M, g, J) is said to be a cohomological Einstein manifold if $c_1 = a\omega$ for some constant $a \in \mathbb{R}$. Of course a compact

Kähler manifold (M, g, J) with the second Betti number $b_2(M) = 1$ is cohomologically Einstein. Finally (M, g, J) is cohomologically Einstein with constant scalar curvature if and only if it is Einstein.

14 Theorem. *Let (M, g, J) and (M', g', J') be two compact Kähler manifolds with $8 \leq q = \dim_{\mathbb{C}} M \leq 51$. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M', g', J') has constant holomorphic sectional curvature c' if and only if (M, g, J) has constant holomorphic sectional curvature $c = c'$.*

Proof. We suppose that (M', g', J') has constant holomorphic sectional curvature c. From (11) and (22) we have

$$\int_M \left[\lambda_q \|B\|^2 + \delta_q \left(\|\rho\|^2 - \frac{r^2}{2q}\right)\right] v_g = \epsilon_q \left(\int_{M'} r'^2 v_{g'} - \int_M r^2 v_g\right) \tag{24}$$

where

$$\lambda_q = 2(2q - 15),$$
$$\delta_q = \frac{-4q^2 + 204q + 120}{q + 2}$$

and

$$\epsilon_q = \frac{10q^4 + 88q^3 + 292q^2 + 342q + 60}{q(q+1)(q+2)}.$$

We remark that if $8 \leq q \leq 51$, then $\lambda_q > 0$, $\delta_q > 0$ and $\epsilon_q > 0$. Hence, by (16), (24) gives $B = 0$ and $\|\rho\|^2 = r^2/2q$. Consequently, (M, g) has constant holomorphic sectional curvature c and, since $r = r' = q(q+1)c'$, $c = c'$. $\boxed{QED}$

We denote by $(\mathbb{CP}^q(c), g_0, J_0)$ the complex projective space with the Study-Fubini metric of constant holomorphic sectional curvature $c > 0$.

15 Corollary. *Let (M, g, J) be a compact Kähler manifold with $8 \leq q \leq 51$. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{\mathbb{CP}^q(c)})$, then (M, g, J) is holomorphically isometric to $(\mathbb{CP}^q(c), g_0, J_0)$.*

16 Theorem. *Let (M, g, J) and (M', g', J') be two compact cohomological Einstein Kähler manifolds, with $q \geq 8$. If $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, then (M', g', J') has constant holomorphic sectional curvature c' if and only if (M, g, J) has constant holomorphic sectional curvature $c = c'$.*

Proof. Since (M, g) and (M', g') are cohomological Einstein

$$c_1 = a\omega, \qquad c_1' = a'\omega',$$

then (see [9, 10] for details)

$$a = \frac{1}{8\pi q \,\mathrm{vol}(M, g)} \int_M r v_g\,, \qquad a' = \frac{1}{8\pi q' \,\mathrm{vol}(M', g')} \int_{M'} r' v_{g'}$$

and

$$\int_M \left(r^2 - 2\|\rho\|^2\right) v_g = 16q(q-1)a^2\pi^2 \operatorname{vol}(M,g),$$
$$\int_{M'} \left(r'^2 - 2\|\rho'\|^2\right) v_{g'} = 16q'(q'-1)a'^2\pi^2 \operatorname{vol}(M',g').$$

By (8)-(10) we obtain $a = a'$ and, consequently,

$$\int_M \left(r^2 - 2\|\rho^2\|^2\right) v_g = \int_{M'} \left(r'^2 - 2\|\rho'^2\|^2\right) v_{g'}. \tag{25}$$

Now we suppose that (M', g', J') has constant holomorphic sectional curvature c. From (11) and (25) we have

$$\lambda_q \int_M \left(\|R\|^2 - \frac{2}{q(q+1)} r^2\right) v_g = \mu_q \left(\int_{M'} r'^2 v_{g'} - \int_M r^2 v_g\right), \tag{26}$$

where

$$\mu_q = \frac{8q^3 + 158q^2 + 158q - 60}{q(q+1)}.$$

Since $\lambda_q > 0$ and $\mu_q > 0$ for $q \geq 8$, (16) and (26) yield

$$\|R\|^2 = \frac{2}{q(q+1)} r^2$$

and hence (M, g, J) has constant holomorphic sectional curvature c. This is equal to c' because $r = r' = q(q+1)c'$. $\boxed{QED}$

17 Remark. By the same proof of Theorem 6, replacing the inequality (13) with the corresponding inequality (23) for Kähler manifolds, we can prove that, if (M, g, J) and (M', g', J') are two compact Einstein Kähler manifolds with $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, then (M', g', J') has constant holomorphic sectional curvature c' if and only if (M, g, J) has constant holomorphic sectional curvature $c = c'$.

Moreover Proposition 5 and Remark 17 give

18 Theorem. *Let (M, g, J) and (M', g', J') be two compact Kähler manifolds with $q = 2$, $\chi(M) \leq \chi(M')$ and (M', g', J') having constant holomorphic sectional curvature c'. If $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, then (M, g, J) has constant holomorphic sectional curvature $c = c'$.*

19 Theorem. *Let (M, g, J) and (M', g', J') be two compact Bochner-Kähler manifolds, with $2 \leq q \leq 51$. If $\operatorname{Spec}(J_M) = \operatorname{Spec}(J_{M'})$, then (M', g', J') has constant holomorphic sectional curvature c' if and only if (M, g, J) has constant holomorphic sectional curvature $c = c'$.*

Proof. The result readily follows from (24) and (16) taking into account that $B = B' = 0$ and that δ_q and ϵ_q are positive for $2 \leq q \leq 51$. $\boxed{QED}$

By Theorems 16, 18, 19, Remark 17 and recalling that $\chi(\mathbb{CP}^2) = 3$ (see [3]), we have

20 Corollary. *Let* (M, g, J) *be a compact Kähler manifold. Assume that* $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{\mathbb{CP}^q(c)})$. *If one of the following cases occurs:*

(a) (M, g, J) *cohomologically Einstein and* $q \geq 8$,

(b) (M, g, J) *Einstein,*

(c) $q = 2$ *and* $\chi(M) \leq 3$,

(d) (M, g, J) *Bochner-Kähler and* $2 \leq q \leq 51$,

then (M, g, J) *is holomorphically isometric to* $(\mathbb{CP}^q, g_0, J_0)$.

4 Spectral geometry of J_M for quaternionic Kähler manifolds

Let (M, g) be a $4q$-dimensional Riemannian manifold ($q = \dim_{\mathbb{H}} M \geq 1$) whose holonomy group is contained in $Sp(n) \cdot Sp(1)$. For $q \geq 2$, (M, g) is called a quaternionic Kähler manifold. In this case, by a well-known Alekseevskii result [1], (M, g) is Einstein. Since $Sp(1) \cdot Sp(1) = SO(4)$, a stronger definition of quaternionic Kähler manifold is needed in dimension 4. A 4-dimensional manifold is said to be a quaternionic Kähler manifold if it is Einstein and self-dual with non-zero scalar curvature [6].

Further

$$\|R\|^2 \geq \frac{5q + 1}{4q(q + 2)^2} r^2 \tag{27}$$

with the equality sign valid if and only if the manifold (M, g) has constant quaternionic sectional curvature [12].

21 Theorem. *Let* (M, g) *and* (M', g') *be two compact quaternionic Kähler manifold. If* $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$, *then* (M', g') *has constant quaternionic sectional curvature* c' *if and only if* (M, g) *has constant quaternionic sectional curvature* $c = c'$.

PROOF. Since the manifolds (M, g) and (M', g') are Einstein with $q = q'$, Theorem 1 gives

$$\int_M \left[2(4q-15)\|R\|^2 + \frac{20q^2+58q+45}{q} r^2 \right] v_g = \int_{M'} \left[2(4q-15)\|R'\|^2 + \frac{20q^2+58q+45}{q} r'^2 \right] v_{g'},$$

that is, since $r = r' = constant$,

$$\int_M \|R\|^2 v_g = \int_{M'} \|R'\|^2 v_{g'}.$$

This and (27) give the required result since for constant quaternionic sectional curvature we have $r = r' = 4q(q+2)c'$. $\boxed{QED}$

Now, by $(\mathbb{HP}^q(c), g_0)$ we denote the quaternionic projective space with constant quaternionic sectional curvature $c > 0$. By Theorem 21 we get

22 Corollary. *Let (M, g) be a compact quaternionic Kähler manifold. If* $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{I_{\mathbb{HP}^q(c)}})$*, then (M, g) is isometric to $(\mathbb{HP}^q(c), g_0)$.*

In dimension four we get the following result.

23 Theorem. *Let (M, g) be a compact 4-dimensional self-dual Riemannian manifold and (M', g') be a compact quaternionic Kähler manifold with positive scalar curvature and $\chi(M') \geq \chi(M)$. If* $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$*, then (M, g) and (M', g') are isometric.*

PROOF. By (8) $\dim M' = 4$, then (M', g') is Einstein and self-dual. Since the two only 4-dimensional self-dual Einstein manifolds with positive scalar curvature are S^4 and $\mathbb{CP}^2$ endowed with the standard metrics ([3], page 275), we get that (M', g') is isometric either to (S^4, g_0) or to $(\mathbb{CP}^2, g_0)$.

By Proposition 5 we deduce that (M, g) is Einstein. Furthermore, by (9) and (10), (M, g) has (constant) scalar curvature $r = r' > 0$ and hence (M, g) is also isometric either to (S^4, g_0) or to $(\mathbb{CP}^2, g_0)$. Taking into account that I_{S^4} is unstable while $I_{\mathbb{CP}^2}$ is stable [13], $\mathrm{Spec}(J_M) = \mathrm{Spec}(J_{M'})$ yields the required result. $\boxed{QED}$

Acknowledgements. The Author thanks Prof. Domenico Perrone for his scientific support and the helpful discussions on this paper.

References

[1] D. V. Alekseevskii: *Riemannian spaces with exceptional holonomy groups*, Functional Anal. Appl., **2** (1968), 97–105.

[2] M. Berger, P. Gauduchon, E. Mazet: Le Spectre d'une Variété Riemannienne, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Hiedelberg, New York 1971.

[3] A. Besse: Géométrie Riemannienne en dimension 4, Cedic/Fernand Nathan, Paris 1981.

[4] P. Gilkey: *The spectral geometry of real and complex manifolds*, Proc. Sympos. Pure Math., **27** (1975), 265–280.

[5] C. Gherghe, S. Ianus, A. M. Pastore: *CR-manifolds, harmonic maps and stability*, J. Geom., **71** (2001), 42–53.

[6] C. LeBrun: *On complete quaternionic-Kähler manifolds*, Duke Math. J., **63** (1991), 723–743.

[7] J. Milnor: *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A., **51** (1964), 542.

[8] S. Nishikawa, P. Tondeur, L. Vanhecke: *Spectral Geometry for Riemannian Foliations*, Annals of Global Analysis and Geometry, **10** (1992), 291–304.

[9] K. Ogiue: *Generalized scalar curvatures of cohomological Einstein Kähler manifolds*, J. Differential Geom., **10** (1975), 201–205.

[10] D. Perrone: *Cohomological Einstein Kähler manifolds characterized by the spectrum*, Math. Z., **185** (1984), 179–183.

[11] R. T. Smith: *The second variation formula for harmonic mappings*, Proc. Amer. Math. Soc., **47** (1975), 229–236.

[12] F. Tricerri, L. Vanhecke: *Decomposition of a space of curvature tensors on a quaternionic Kähler manifold and spectrum theory*, A Quarterly Journal of Pure and Applied Mathematics, **53** (1979), 163–173.

[13] H. Urakawa: *Calculus of Variations and Harmonic Maps*, American Mathematical Society, Providence, Rhode Island 1993.

[14] H. Urakawa: *Spectral Geometry of the second variation operator of harmonic maps*, Illinois Journal of Mathematics, **33** (1989), 250–267.